

$\sin \theta_{13}$ and neutrino mass matrix with an approximate flavor symmetry and that with SU(3) symmetry

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ABSTRACT: For a neutrino mass matrix with an approximate flavor symmetry where one has near degenerate neutrino mass, it is shown that the tri-bimaximal values for atmospheric angle $\sin^2 \theta_{23} = \frac{1}{2}$ and solar angle $\sin^2 \theta_{12} = \frac{1}{3}$ can be maintained even when the reactor angle $\theta_{13} \neq 0$. For a neutrino mass matrix obtained from an SU(3) symmetry for neutrinos in the mass eigenstate and its breaking, it is shown that the non-zero value of $\sin \theta_{13}$ can be accommodated with a scale which breaks the SU(3) where one has inverted mass hierarchy symmetry. In both cases non zero $\sin \theta_{13}$ implies approximate $\nu_\mu \rightarrow -\nu_\tau$ symmetry instead of $\nu_\mu \rightarrow \nu_\tau$ symmetry.

KEYWORDS: Neutrino-Physics, Flavor Symmetry, Nonzero reactor angle

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1 Introduction

There is compelling evidence that neutrinos change flavor, have non zero masses and that neutrino mass eigenstates are different from weak eigenstates. As such they undergo oscillations. The flavor and mass eigenstates are related by the so called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [1],

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad (1.1)$$

where the matrix U has been parametrized by the Particle Data Group (PDG) as [2]

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \quad (1.2)$$

Here $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$ and P is a diagonal matrix which contains (Majorana) CP violating phases in addition to the (Dirac) CP violating phase δ . θ_{12} , θ_{23} and θ_{13} are respectively known as solar, atmospheric and reactor angles.

Current global fits allow the following ranges for the mass squared differences and mixing angles [2]:

$$\begin{aligned} 7.05 \times 10^{-5} eV^2 &\leq \Delta m_{12}^2 \leq 8.34 \times 10^{-5} eV^2, \\ 0.25 &\leq \sin^2 \theta_{12} \leq 0.37, \\ 2.70 \times 10^{-3} eV^2 &\leq \Delta m_{31}^2 \leq 2.75 \times 10^{-3} eV^2, \\ 0.36 &\leq \sin^2 \theta_{23} \leq 0.67, \end{aligned} \quad (1.3)$$

with the following best fit (BF) values

$$\Delta m_{12}^2 = 7.65 \times 10^{-5} eV^2, \quad \sin^2 \theta_{12} = 0.304, \quad \Delta m_{31}^2 = 2.40 \times 10^{-3} eV^2, \quad \sin^2 \theta_{23} = 0.5. \quad (1.4)$$

Recent results from T2K collaboration [3] and MINOS indicate a relatively large θ_{13} and when combined with the global fit gives [4]

$$\sin^2 \theta_{13} = 0.025 \pm 0.007. \quad (1.5)$$

As we shall see it is interesting on its own right to consider non-zero value for $\sin^2 \theta_{13}$ in the above range.

It is well known that the best fit values given in Eq. (1.4) are consistent with the so called tri-bimaximal (TB) mixing [5] corresponding to

$$\sin^2 \theta_{12} = \frac{1}{3}, \quad \sin^2 \theta_{23} = \frac{1}{2}, \quad \sin^2 \theta_{13} = 0. \quad (1.6)$$

We have three mixing angles, $\theta_{12} = \theta_s$, $\theta_{23} = \theta_a$, $\theta_{13} = \theta_r$ and two squared mass differences $\Delta m_{21}^2 = m_2^2 - m_1^2 = \delta m^2$, $\Delta m^2 = m_3^2 - \frac{m_2^2 + m_1^2}{2}$ where $\Delta m^2 > 0$ for normal hierarchy ($m_1 < m_2 \leq m_3$) and < 0 for inverted hierarchy ($m_1 \simeq m_2 \gg m_3$). For degenerate case ($m_1 \simeq m_2 \simeq m_3$), and one can write neutrino mass matrix as

$$M_\nu = m_0 I + \delta M_\nu \quad (1.7)$$

where $\delta M_\nu \ll m_0$

2 Approximate flavor symmetry and diagonalization of neutrino mass matrix

For degenerate case a particularly attractive Majorana neutrino mass matrix, which preserves flavor with small perturbations violating it in off-diagonal matrix elements is given by [6, 7]

$$M_\nu = m_0 \begin{pmatrix} 1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & 1 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & 1 \end{pmatrix} = m_0 I + \delta M_\nu, \quad (2.1)$$

with $\epsilon_{ij} \ll 1$. It may be noted that off-diagonal elements were previously considered [8] but in different contexts. Recently, a possibility has been discussed [9] which allows the extension of TB mixing, to have a non-zero value of θ_{13} , preserving at the same time the predictions for the TB solar angle [$\sin^2 \theta_{12} = \frac{1}{3}$] and the maximal atmospheric angle [$\sin^2 \theta_{23} = \frac{1}{2}$]. For our discussion it is convenient to state various symmetries and/or conditions on M_ν which lead to $\theta_{13} = 0$ and T.B. mixing. In an obvious notation if one has 2-3 symmetry i.e. $(M_\nu)_{22} = (M_\nu)_{33}$ and $(M_\nu)_{12} = (M_\nu)_{13}$, then $\theta_{13} = 0$ and $\sin^2 \theta_{23} = 1/2$. If further $(M_\nu)_{11} + (M_\nu)_{12} = (M_\nu)_{22} + (M_\nu)_{23}$, then we have T.B. mixing given in Eq. (1.6).

In this paper we explore the above possibility for the diagonalization of the neutrino mass matrix given in Eq. (1.7). In spite of its attractiveness, its diagonalization is in conflict with the neutrino data. It is instructive to show it as it would provide us a guidance for possible modification of M_ν in Eq. (2.1) to obtain agreement with the experimental data. The diagonalization of M_ν (we need to consider the diagonalization of δM_ν as $m_0 I$ commutes with any diagonalizing matrix) give among others the following relation [10]

$$\epsilon_2 = \frac{\cos^2 \theta_s - \tan^2 \theta_r}{\sin^2 \theta_s - \tan^2 \theta_r} \epsilon_1, \quad \epsilon_1 + \epsilon_2 + \epsilon_3 = 0 \quad (2.2)$$

Now

$$m_2 = m_0(1 + \epsilon_2), \quad m_1 = m_0(1 + \epsilon_1), \quad m_3 = m_0(1 - \epsilon_1 - \epsilon_2) \quad (2.3)$$

The first of relations (2.2) give

$$\epsilon_+(1 - 2 \tan^2 \theta_r) = \epsilon_- \cos 2\theta_s \quad (2.4)$$

while

$$\Delta m_{12}^2 = 4m_0^2 \epsilon_- \quad (2.5a)$$

$$\Delta m^2 = -6m_0^2 \epsilon_+ \quad (2.5b)$$

where

$$\epsilon_+ = \frac{\epsilon_2 + \epsilon_1}{2}, \quad \epsilon_- = \frac{\epsilon_2 - \epsilon_1}{2} \quad (2.6)$$

We require $\epsilon_- \ll \epsilon_+$ and the relation (2.4) then implies that $\tan^2 \theta_r \simeq 1/2$, contrary to the experimental data. The relation (2.2) is the consequence of $\det|\delta M_\nu| = 0$. To avoid this, the simplest extension is that

$$\delta M_\nu = \begin{pmatrix} a & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & 0 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & 0 \end{pmatrix} \quad (2.7)$$

where $a \ll 1$. Then the relations in Eq. (2.2) are replaced by

$$\begin{aligned} a &= -\epsilon_1(\tan^2 \theta_r - \cos^2 \theta_s) - \epsilon_2(\tan^2 \theta_r - \sin^2 \theta_s) \\ &= \epsilon_+(1 - 2 \tan^2 \theta_r) - \epsilon_- \cos 2\theta_s \end{aligned} \quad (2.8)$$

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = a \quad (2.9)$$

Further while the relation (2.5a) remains the same but (2.5b) is changed to

$$\begin{aligned} \Delta m^2 &= m_0^2[2a - 6\epsilon_+] \\ &= -4m_0^2[\epsilon_+(1 + \frac{1}{2} \tan^2 \theta_r) + \frac{1}{2} \epsilon_- \cos 2\theta_s] \end{aligned} \quad (2.10a)$$

On using Eq. (2.5a)

$$|\Delta m^2| = 4m_0^2 \epsilon_+(1 + \tan^2 \theta_r) + \frac{1}{2} \cos \theta_s \Delta m_{12}^2$$

Since $|\Delta m^2| \gg \Delta m_{12}^2$, it follows that

$$|\Delta m^2| \simeq 4m_0^2 \quad (2.10b)$$

Thus from Eqs. (??), (2.10b) and (1.4)

$$\frac{\epsilon_-}{\epsilon_+} = \frac{\Delta m_{12}^2}{|\Delta m^2|} = 3.2 \times 10^{-2} \quad (2.11)$$

The other relations which diagonalization give are

$$\cos 2\theta_a \epsilon_+ + \sin 2\theta_a \sin 2\theta_s \sin 2\theta_r \epsilon_- = 0 \quad (2.12)$$

$$\epsilon_{12} + \epsilon_{13} = \cos \theta_r \sin 2\theta_s (\cos \theta_a - \sin \theta_a) \epsilon_- - 2(\cos \theta_a + \sin \theta_a) \tan \theta_r \epsilon_+ \quad (2.13)$$

$$\epsilon_{12} - \epsilon_{13} = \cos \theta_r \sin 2\theta_s (\cos \theta_a + \sin \theta_a) \epsilon_- - 2(\cos \theta_a - \sin \theta_a) \tan \theta_r \epsilon_+ \quad (2.14)$$

$$\epsilon_{23} = -\sin 2\theta_a (\epsilon_+ + \epsilon_- \cos 2\theta_s) - \epsilon_- \cos 2\theta_a \sin 2\theta_s \quad (2.15)$$

To proceed further it is convenient to use the expansion about the maximum atmospheric angle $\sin^2 \theta_a = \frac{1}{2}$

$$\sin \theta_a = -\frac{1}{\sqrt{2}}(1+t), \quad \cos \theta_a = \frac{1}{\sqrt{2}}(1-t), \quad \sin^2 \theta_a = 0.5 + t \quad (2.16)$$

Then the relations (2.12-2.15) simplify to

$$-2t\epsilon_+ - \sin 2\theta_s \sin \theta_r \epsilon_- = 0 \quad (2.17)$$

$$\epsilon_{12} - \epsilon_{13} = -\sqrt{2}t \cos \theta_r \sin 2\theta_s \epsilon_- + 2\sqrt{2} \tan \theta_r \epsilon_+ \quad (2.18)$$

$$\epsilon_{23} = \epsilon_+ + \epsilon_- \cos 2\theta_s \quad (2.19)$$

$$\begin{aligned} \epsilon_{12} + \epsilon_{13} &= \sqrt{2} \cos \theta_r \sin 2\theta_s \epsilon_- + 2\sqrt{2}t \tan \theta_r \epsilon_+ \\ &= \sqrt{2} \sin 2\theta_s \left(1 - \frac{3}{2} \sin^2 \theta_r\right) \epsilon_- \end{aligned} \quad (2.20)$$

where in the second step we have used Eq. (2.17). Further from Eqs. (2.8) and (2.19)

$$a - \epsilon_{23} = -2\epsilon_+ \tan^2 \theta_r - 2\epsilon_- \cos 2\theta_s \quad (2.21)$$

so that together with Eq. (2.20)

$$\tan 2\theta_s = \frac{-\sqrt{2}(\epsilon_{12} + \epsilon_{13})(1 + \frac{3}{2} \sin^2 \theta_r)}{a - \epsilon_{23} + 2 \tan^2 \theta_r \epsilon_+} \quad (2.22)$$

We note that if $\nu_\mu \leftrightarrow \nu_\tau$ ($2 \leftrightarrow 3$) symmetry is imposed so that $\epsilon_{12} = \epsilon_{13}$, $t \rightarrow 0$, $\theta_r \rightarrow 0$, the relations (2.17), (2.18) are identically satisfied. Further if $a - \epsilon_{23} = -\epsilon_{12}$, then Eq. (2.22) gives $\tan 2\theta_s = 2\sqrt{2}$ i.e. the T.B. solar angle.

However if $\sin \theta_r \neq 0$, Eq. (2.17) implies that

$$\begin{aligned} \sin \theta_r &= -\frac{2t}{\sin 2\theta_s} \frac{\epsilon_+}{\epsilon_-} \\ &\simeq -\frac{1}{\sqrt{2}} t \times 10^2 \end{aligned} \quad (2.23)$$

on using Eq. (2.11) and the T.B value of θ_s . This can accommodate any finite value of $\sin \theta_r$ for extremely small deviation from the maximal atmospheric angle $\sin^2 \theta_a = \frac{1}{2}$; for

$\sin \theta_r \leq 1$, $|t| < \sqrt{2} \times 10^{-2}$. For $\sin^2 \theta_r =$ give in Eq. (1.5)[$\sin \theta_r \simeq \sqrt{2} \times 10^{-1}$], it requires only $t \sim -2 \times 10^{-3}$. It follows from Eq. (2.20) that T. B. solar angle is obtained i.e. $\tan 2\theta_s = 2\sqrt{2}$, if

$$a - \epsilon_{23} + 2 \tan^2 \theta_r \epsilon_+ = -\frac{1}{2}(\epsilon_{12} + \epsilon_{13})(1 + \frac{3}{2} \sin^2 \theta_r) \quad (2.24)$$

It remains to determine the parameters in the M_ν given in Eq. (2.7). It follows from Eqs. (2.10b), (2.19), (2.20) and (2.21)

$$\epsilon_{23} \simeq a \simeq \epsilon_+ = \frac{1}{4} \frac{|\Delta m^2|}{m_0^2} \quad (2.25)$$

$$\frac{\epsilon_{12} + \epsilon_{13}}{\epsilon_+} \simeq \sqrt{2} \sin 2\theta_s \frac{\epsilon_-}{\epsilon_+} \simeq 4 \times 10^{-2} \quad (2.26)$$

On the other hand from Eqs. (1.5) and (2.18)

$$\frac{\epsilon_{12} - \epsilon_{13}}{\epsilon_+} \simeq 2\sqrt{2} \tan \theta_r \simeq 4 \times 10^{-1} \quad (2.27)$$

Thus it is possible to have T.B. solar angle $\sin^2 \theta_s = \frac{1}{3}$ and almost maximal atmospheric angle $\sin^2 \theta_a \simeq \frac{1}{2}$ and non zero $\sin^2 \theta_r$ but at the cost of $\nu_\mu \rightarrow \nu_\tau$ symmetry as the relations (2.26) and (2.27) would imply. Finally the oscillation data gives only a lower bound on the heaviest of the neutrino mass $m_h \geq |\Delta m^2| > 0.05\text{eV}$ but cannot fix it. However m_0 is further constrained by WMAP data, $\sum m_i < (0.4 - 0.7)\text{eV}$. Taking $m_0 \simeq 0.1\text{eV}$, we get from Eqs. (1.4), (2.25), (2.26) and (2.27) that

$$\begin{aligned} \epsilon_{23} \simeq a \simeq \epsilon_+ &\simeq 6 \times 10^{-2} \\ \epsilon_{12} + \epsilon_{13} &\simeq 2.4 \times 10^{-3} \\ \epsilon_{12} - \epsilon_{13} &\simeq 2.4 \times 10^{-2} \end{aligned}$$

i.e.

$$\epsilon_{12} \simeq -\epsilon_{13} \simeq 10^{-2} \quad (2.28)$$

All the above values are consistent with small perturbations (at least an order of magnitude smaller) to $M_\nu = m_0 I$, I being the unit matrix. It is important to note that Eq. (2.28) implies approximate $\nu_\mu \rightarrow -\nu_\tau$ symmetry (instead of $\nu_\mu \rightarrow \nu_\tau$ symmetry) which is valid by about 10%.

3 SU(3) and non-zero $\sin \theta_{13}$

We now consider the neutrino mass matrix in the flavor basis which was obtained by assuming that the light neutrinos form a triplet in a global SU(3) in the mass eigenstate basis, which is broken in the direction $(-a\lambda_3 + \frac{b}{\sqrt{3}}\lambda_3)$, λ_3 and λ_8 being the only diagonal Gell-Mann matrices in SU(3)[11]. In this case one has inverted mass hierarchy

The resulting mass matrix, assuming TB solar angle and the maximal atmospheric angle but allowing $\sin\theta_{13} \neq 0$, in the flavor basis is

$$M_\nu = m_0 I + M, \quad (3.1)$$

where (neglecting terms of order $a \sin\theta_{13}$, $b \sin^2\theta_{13}$)

$$M = \frac{b-a}{2} \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 1 \\ 0 & 1 & -1/3 \end{pmatrix} - \frac{a}{2} \begin{pmatrix} 0 & -4/3 & -4/3 \\ -4/3 & 0 & -1/3 \\ -4/3 & -1/3 & 0 \end{pmatrix} + \frac{b}{\sqrt{2}} \sin\theta_{13} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (3.2)$$

where [10]

$$\begin{aligned} \frac{a}{b} &= \frac{1}{2} \frac{\Delta m_{12}^2}{|\Delta m_{32}^2|} \simeq 1.5 \times 10^{-2}, \\ \frac{b}{m_0} &= \frac{1}{2} \frac{\Delta m_{23}^2}{m_0^2} \simeq 1.2 \times 10^{-3} \frac{eV^2}{m_0^2} \simeq 10^{-1}, \end{aligned} \quad (3.3)$$

if we take $m_0 \simeq 0.1 eV$. The dominant part of M , namely the first term in Eq. (3.2) was also obtained from other considerations [7, 12], but this by itself can not lift the degeneracy between m_1 and m_2 . On the other hand SU(3) considerations naturally lift the degeneracy as a consequence of SU(3) symmetry breaking with small symmetry breaking parameters [11]. The above matrix allows non-zero $\sin\theta_{13}$ and it is interesting to note that its value given in Eq. (2.27) is of order $\sqrt{2} \frac{b}{m_0}$, the dominant scale in M given in Eq. (3.2) which breaks the exact SU(3) symmetry. It is interesting to note that the third term in Eq. (3.2) has $\nu_\mu \leftrightarrow -\nu_\tau$ symmetry and since it dominates over the second term (which has $\nu_\mu \leftrightarrow \nu_\tau$ symmetry), one may conclude that in this case also one has $\nu_\mu \rightarrow -\nu_\tau$ symmetry within 10%.

4 Summary and Conclusion

We have considered two models of approximate flavor symmetry. In the first case one has near degenerate neutrino mass. It is shown that it is possible to have nonzero solar angle while preserving the T.B solar angle $\sin^2\theta_s = \frac{1}{3}$ and near maximal atmospheric angle $\sin^2\theta_a \simeq 0.5$. In the second case we start from a generated SU(3) symmetry for the neutrino mass eigenstates which is broken along the direction $(-a\lambda_3 + \frac{b}{3}\lambda_8)$ with $a \ll b$. Here one has inverted mass hierarchy. Again it is possible to have non zero $\sin\theta_r$ which is of the order of dominant scale while breaks the exact SU(3) symmetry. In both cases non-zero $\sin\theta_r$ implies $\nu_\mu \rightarrow -\nu_\tau$ symmetry (with 10%) rather than that $\nu_\mu \rightarrow \nu_\tau$ symmetry.

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